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# Slender Wing Theory for Nonuniform Stream

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M. Hanin* and A. Barsony-Nagy†
*Technion—Israel Institute of Technology, Haifa, Israel*

**A method is developed for calculating the flow and the lift force on a slender pointed wing in a parallel stream whose velocity varies in the vertical direction. The Euler equations of flow are linearized for small disturbances to the nonuniform stream, and an integral equation for lift distribution on the wing is deduced, using the simplifying assumption of small aspect ratio. The integral equation is solved by an analytic expansion. Solutions calculated for several stream velocity profiles show how the lift varies with the velocity ratio of the stream and with the ratio of the wing span to the vertical extent of the nonuniform region.**

## Introduction

THE aerodynamics of wing in a nonuniform parallel stream is of practical interest since it relates to flight in a sheared wind, a jet, or wake. From a more theoretical point of view, it is of interest to know how the flow and the aerodynamic forces behave when the usual assumption of uniform upstream velocity is removed. However, the existing literature on three-dimensional wings in nonuniform stream is not extensive. Karman and Tsien[1] considered a lifting line approximation and applied a Fourier transform method. Dowell,[2] Ventres,[3] and Williams et al.[4] obtained numerical lifting-surface solutions for panels and rectangular wings in a shear layer simulating a turbulent boundary layer. Homentcovschi and Barsony-Nagy[5] formulated a lifting-surface integral equation for general stream velocity profiles.

In the present study, a slender wing in a nonuniform stream is considered. The wing has a low aspect ratio, a pointed planform, no thickness, and is set at an angle of attack. The stream is parallel and its velocity $U$ varies with the vertical distance $z$ for the wing plane, so that the stream velocity profile is given by $U(z)$. The velocity $U$ is supposed to attain finite values as the distance $z$ tends to infinity above or below the wing. It is assumed that the flow is incompressible, since for a slender wing the effects of Mach number are negligible at subsonic and low supersonic speeds. In the analysis, the Euler equations of flow are linearized for small perturbations of the nonuniform stream and then are simplified for a slender wing. Thus, the streamwise second-order derivatives are neglected in comparison with the lateral second-order derivatives, and the flow problem becomes two-dimensional in the cross flow planes. An integral equation for the lift distribution on the wing is then deduced with the aid of a Fourier transformation. The integral equation is solved by an analytic expansion method. Solutions are obtained for several stream velocity profiles, and a brief parametric study is made of the effects of stream nonuniformity on the lift.

## Equations of Flow

The flow due to the wing is regarded as a small perturbation to the nonuniform stream velocity $U(z)$ parallel to the $x$ axis. The perturbations of the velocity components and pressure are denoted by $u,v,w,$ and $p$. Linearizing the Euler momentum and continuity equations of steady incompressible inviscid flow, we have

$$U(z)u_x + U'(z)w + p_x/\rho = 0 \tag{1a}$$

$$U(z)v_x + p_y/\rho = 0 \tag{1b}$$

$$U(z)w_x + p_z/\rho = 0 \tag{1c}$$

$$u_x + v_y + w_z = 0 \tag{2}$$

where the subscript derivative notation is used.

A single differential equation for the upward velocity component $w$ is obtained by eliminating the other variables from Eqs. (1) and (2). An integration with respect to $x$, taking into account that $w$ vanishes at infinity upstream, gives then for a slender wing

$$U(z)(w_{yy} + w_{zz}) - U''(z)w = 0 \tag{3}$$

Here a term $w_{xx}$ has been neglected in accordance with the slender wing approximation. The resulting equation is two-dimensional in the cross flow planes $(y,z)$ with the $x$ coordinate serving as a parameter.

The wing is situated near the $z=0$ plane. Denoting by $\alpha(x,y)$ the local angle of attack, the linearized boundary condition on the wing is

$$w(x,y,0) = -U(0)\alpha(x,y) \tag{4}$$

The derivative $w_z$ is discontinuous across the wing, and its jump is related to the pressure jump. In fact, Eqs. (1a), (1b), and (2) give

$$U(z)w_{xz} - U'(z)w_x = p_{yy}/\rho \tag{5}$$

where again a term $p_{xx}$ has been neglected. Taking this equation at the wing on each side and forming the difference, and then integrating with respect to $x$, we obtain a relation between the jump of $w_z$ across the wing and the chordwise integral of the pressure load $L(x,y)$,

$$w_z(x,y,+0) - w_z(x,y,-0) = -\tfrac{1}{2}U(0)L_{yy}(x,y) \tag{6}$$

where

$$L(x,y) = \frac{2}{\rho U^2(0)} \int_{x_\ell(y)}^{x} [p(\xi,y,-0) - p(\xi,y,+0)]\,d\xi \tag{7}$$

and $x_\ell(y)$ denotes the $x$ coordinate of the leading edge. The function $L(x,y)$ gives the spanwise lift distribution of the wing, from the apex to the $x$ cross section.

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*Professor, Dept. of Aeronautical Engineering. Member AIAA.

†Instructor, Dept. of Aeronautical Engineering.

## Integral Equation for Lift Distribution

By solving the differential equation (3) with the jump condition (6), and then applying the boundary condition (4), an integral equation will be obtained for the lift distribution $L(x,y)$.

To solve Eq. (3), a Fourier transformation is made with respect to $y$. Any function $f(x,y,z)$ is related to its transform $\tilde{f}(x,k,z)$ by

$$f(x,y,z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\tilde{f}(x,k,z)e^{iyk}\mathrm{d}k \tag{8}$$

Equations (3), (6), and (4) give

$$\tilde{w}_{zz}-\left[k^2+\frac{U''(z)}{U(z)}\right]\tilde{w}=0 \tag{9}$$

$$\tilde{w}_z(z,k,+0)-\tilde{w}_z(x,k,-0)=\tfrac{1}{2}U(0)k^2\tilde{L}(x,k) \tag{10}$$

$$\tilde{w}(x,k,+0)-\tilde{w}(x,k,-0)=0 \tag{11}$$

Since the perturbations vanish at infinite lateral distances from the wing, we also have

$$\tilde{w}(x,k,\pm\infty)=0 \tag{12}$$

The transform solution $\tilde{w}$ can be represented in the form

$$\tilde{w}=\tfrac{1}{2}U(0)\,\tilde{W}(k,z)\tilde{L}(x,k) \tag{13}$$

where $\tilde{W}$ is the solution of the differential equation (9) satisfying the conditions

$$\tilde{W}_z(k,+0)-\tilde{W}_z(k,-0)=k^2 \tag{14a}$$

$$\tilde{W}(k,+0)-\tilde{W}(k,-0)=0 \tag{14b}$$

$$\tilde{W}(k,\pm\infty)=0 \tag{14c}$$

Once the function $\tilde{W}(k,z)$ and its inverse W(y,z) are determined, the upward velocity $w$ is obtained by applying the convolution theorem to Eq. (13), which gives

$$w(x,y,z)=\frac{U(0)}{2\sqrt{2\pi}}\int_{-s(x)}^{+s(x)}L(x,\eta)\,W(y-\eta,z)\mathrm{d}\eta \tag{15}$$

The boundary condition (4) on the wing yields then an integral equation for the lift distribution $L(x,y)$,

$$\int_{-s(x)}^{+s(x)}L(x,\eta)\,W(y-\eta,0)\mathrm{d}\eta=-2\sqrt{2\pi}\alpha(x,y) \tag{16}$$

where $s(x)$ denotes the local semispan. The kernel $W$, as well as its transform $\tilde{W}$, depends only upon the stream velocity profile $U(z)$ and is independent of the wing planform, camber, or angle of attack.

The transform solution $\tilde{W}(k,z)$ coincides (except for a factor $k^2$) with the fundamental transform solution of Weissinger's theory of two-dimensional airfoils.[6] In fact, the equations and boundary conditions that define the two solutions have the same form, though our $k$ is related to the spanwise variable $y$ instead of the chordwise variable $x$ of airfoil theory. Weissinger investigated the transform solution and its inverse by means of expansions for small $k$ and large $k$, as was done earlier by Lighthill in his study of source flow in nonuniform stream.[7]

An asymptotic expansion for large $k$ showed that

$$\tilde{W}(k,0)=-\tfrac{1}{2}|k|+\frac{U''(0)}{4U(0)}\frac{1}{|k|}+O(k^{-3}) \tag{17}$$

Inverting, and using the fact that $\tilde{W}$ is a continuous function of $k$, we find that the kernel $W$ has the form

$$W(y-\eta,0)=\frac{1}{\sqrt{2\pi}}\left[\frac{1}{(y-\eta)^2}-\sigma\ln|(y-\eta)\sqrt{\sigma}|-N(y-\eta)\right] \tag{18}$$

where $N(y-\eta)$ is a continuous bounded function, and $\sigma$ denotes for brevity

$$\sigma=\frac{U''(0)}{2U(0)} \tag{19}$$

The slender wing integral equation (16) is then

$$\int_{-s(x)}^{+s(x)}\left[(y-\eta)^{-2}-\sigma\ln|\sigma^{1/2}(y-\eta)|-N(y-\eta)\right]L(x,\eta)\mathrm{d}\eta$$
$$=-4\pi\alpha(x,y) \tag{20}$$

In the first term the usual Cauchy principal value must be taken. The second and the third terms are due to the stream nonuniformity.

The bounded part $N$ of the kernel can be computed by solving numerically the transform differential equation (9) with the boundary conditions [(Eq. (14)] for $\tilde{W}$, inverting by means of Eq. (8) at $z=0$, and subtracting the singular terms as indicated in Eq. (18). A more accurate way is to subtract before inverting, since then the singularities are avoided. It can be shown that

$$N(y-\eta)=-\sigma[K_0(|\sigma^{1/2}(y-\eta)|)+\ln|\sigma^{1/2}(y-\eta)|)]$$
$$-\int_0^{\infty}\tilde{F}(k)\cos k(y-\eta)\,\mathrm{d}k \tag{21}$$

where $K_0$ is the Bessel function, and

$$\tilde{F}(k)=2\tilde{W}(k,0)+|k|-\sigma(k^2+|\sigma|)^{-1/2} \tag{22}$$

The function $N$ is even. It is determined by the stream velocity profile only, and does not depend upon wing geometry or the angle of attack.

## Method of Solution

To solve the integral equation (20), an expansion in Fourier series can be used. The variables $y$ and $\eta$ on the wing are changed to $\phi$ and $\theta$ by

$$y=s(x)\cos\phi \qquad (0\leq\phi\leq\pi)$$
$$\eta=s(x)\cos\theta \qquad (0\leq\theta\leq\pi) \tag{23}$$

The lift distribution is expanded by setting

$$L(x,\eta)=s(x)\sum_{m=1}^{\infty}L_m(x)\sin(m\theta) \tag{24}$$

which insures that $L$ will vanish at the leading edge. In Eq. (20), the contributions of the two singular terms of the kernel can now be expressed analytically. For the second term we use the formula[8]

$$\ln|\cos\phi-\cos\theta|=-\ln 2-2\sum_{j=1}^{\infty}\frac{1}{j}\cos(j\phi)\cos(j\theta) \tag{25}$$

A set of algebraic equations for the coefficients $L_m(x)$ of the lift distribution is now obtained by integrating Eq. (20) so as to separate the Fourier components. The resulting equations are

$$\left(1+\lambda \ln|\lambda| - \tfrac{1}{2}\lambda\right)L_1 + \tfrac{1}{2}\lambda L_3 + \sum_{m=1}^{\infty} N_{m,1}L_m = \alpha_1 \qquad (26a)$$

$$\left(2-\frac{4}{3}\lambda\right)L_2 + \frac{1}{3}\lambda L_4 + \sum_{m=1}^{\infty} N_{m,2}L_m = \alpha_2 \qquad (26b)$$

$$\left(n-\frac{2n}{n^2-1}\lambda\right)L_n + \frac{1}{n-1}\lambda L_{n-2} + \frac{1}{n+1}\lambda L_{n+2} + \sum_{m=1}^{\infty} N_{m,n}L_m = \alpha_n \qquad (n=3,4,\ldots) \qquad (26c)$$

where we denote

$$\lambda = \frac{1}{4}\sigma s^2(x) = \frac{U''(0)s^2(x)}{8U(0)} \qquad (27)$$

$$\alpha_n = \frac{8}{\pi}\int_0^{\pi} \alpha(x,s(x)\cos\phi)\sin(n\phi)\sin\phi \, d\phi \qquad (28)$$

$$N_{m,n} = \frac{2}{\pi^2}s^2(x)\int_0^{\pi}\int_0^{\pi} N[s(x)(\cos\phi-\cos\theta)]\sin(m\theta) \cdot \sin(n\phi)\sin\theta\sin\phi \, d\theta \, d\phi \qquad (29)$$

The integrals $N_{m,n}$ can be computed numerically without difficulty since the function $N$ is bounded on the wing. As $N$ is an even function, we have

$$N_{m,n} = 0 \text{ for } m+n = 3,5,\ldots 2r+1\ldots \qquad (30)$$

Thus the set of Eqs. (26) splits into two sets, one involving the asymmetric terms $L_{1+2r}$ and $\alpha_{1+2r}$ $(r=0,1,2,\ldots)$, and the other involving the antisymmetric terms $L_{2+2r}$ and $\alpha_{2+2r}$. Usually the wing is symmetric with respect to the $y=0$ plane, and then

$$\alpha_{2+2r} = 0, \qquad L_{2+2r} = 0 \qquad (31)$$

The lift $L(x)$ of the wing between the apex and the $x$ cross section is related to the first coefficient $L_1$ only,

$$L(x) = \tfrac{1}{2}\rho U^2(0)\int_{-s(x)}^{+s(x)} L(x,y)\,dy = \frac{\pi}{4}\rho U^2(0)s^2(x)L_1(x) \qquad (32)$$

In solving the set of Eqs. (26), we take a finite number of equations and unknowns. In most cases, three terms of the expansion are sufficient for a good accuracy.

## Examples and Numerical Results

To examine the effects of nonuniform stream on the lift of a slender wing, solutions were calculated for a plane wing ($\alpha=$const) and several stream velocity profiles, representing flight in a jet, in a wake, and in a sheared wind.

The jet stream and wake stream profiles are described by

$$U = U_1 + (U_0 - U_1)e^{-z^2/h^2} \qquad (33)$$

Here the wing location $z=0$ is taken at the symmetry plane of the stream, and $U_0 = U(0)$ is the local stream velocity there. The external velocity is $U_1$ and the vertical extent of the jet or wake is related to $h$.

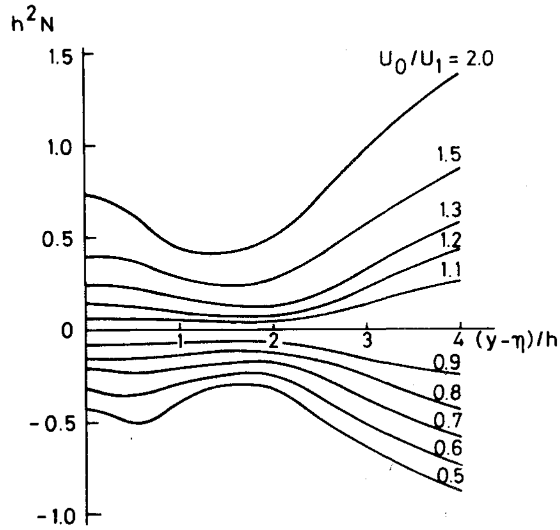


**Fig. 1 The function $N$ for jet stream and wake stream.**

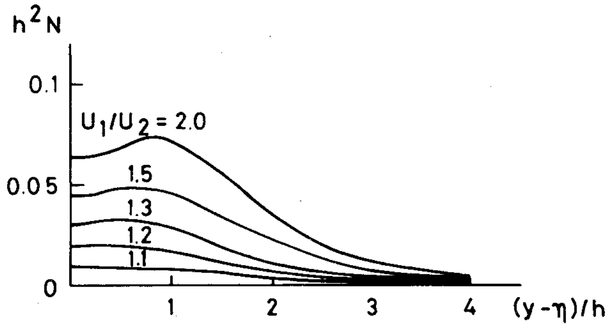


**Fig. 2 The function $N$ for linear sheared stream.**

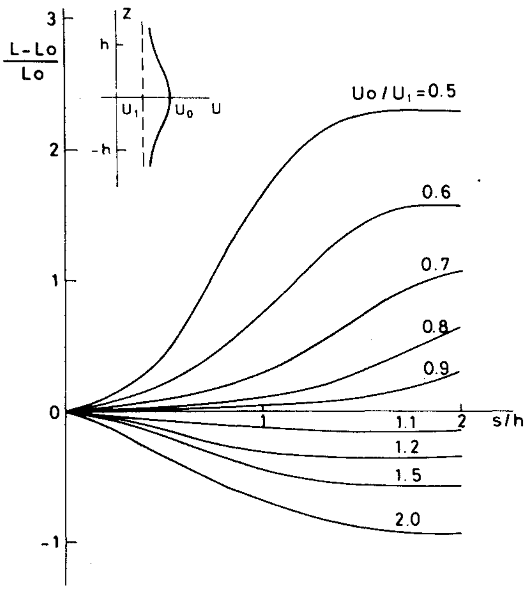


**Fig. 3 Lift increment in jet stream and wake stream.**

The linear shear profile, with the wing placed in the middle plane, is given by

$$U = U_1 \text{ for } z \geq h$$

$$U = \tfrac{1}{2}(U_1+U_2) + \tfrac{1}{2}(U_1+U_2)z/h \text{ for } -h \leq z \leq h$$

$$U = U_2 \text{ for } z \leq -h \qquad (34)$$

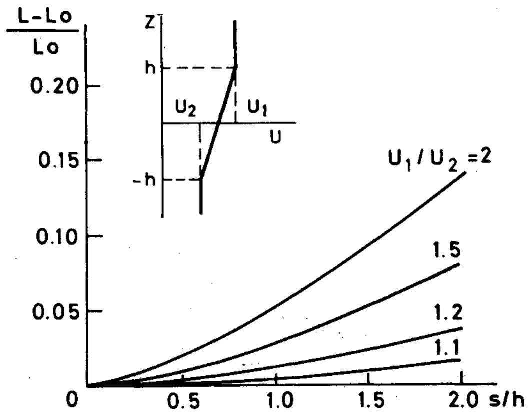


Fig. 4 Lift increment in linear shear stream.

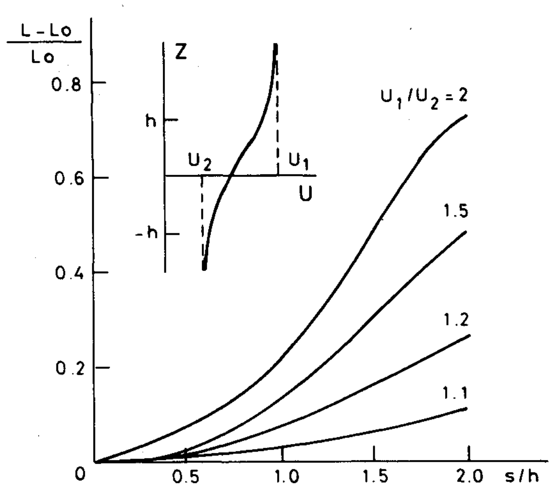


Fig. 5 Lift increment in nonlinear sheared stream.

A nonlinear shear profile was chosen as

$$U = \tfrac{1}{2}(U_1+U_2) + \tfrac{1}{2}(U_1-U_2)\tanh(z-z_0)/h \tag{35}$$

and we took $z_0 = h/3$, so that the wing in this example is placed below the middle plane of the shear layer.

Figures 1 and 2 show the function $N(y-\eta)$ computed by solving numerically Eq. (9) with the conditions of Eq. (14), and inverting as indicated in Eqs. (21) and (22).

The resulting lift increment $(L-L_0)/L_0$ due to stream nonuniformity is shown in Figs. 3-5. Here $L$ is the actual lift as found from Eqs. (26-32), and $L_0$ denotes the lift of the same wing at the same angle of attack in a uniform stream of velocity $U(0)$. Thus

$$(L-L_0)/L_0 = (L_1/\alpha_1) - 1 \tag{36}$$

$$L_0 = \pi\rho U^2(0)s^2(x)\alpha$$

The parameters upon which the relative lift increment depends are the velocity ratio $U_0/U_1$ or $U_1/U_2$, and the wing-to-stream scale ratio $s/h$, where $s$ is the semispan of the wing, and $h$ represents the vertical extent of stream nonuniformity.

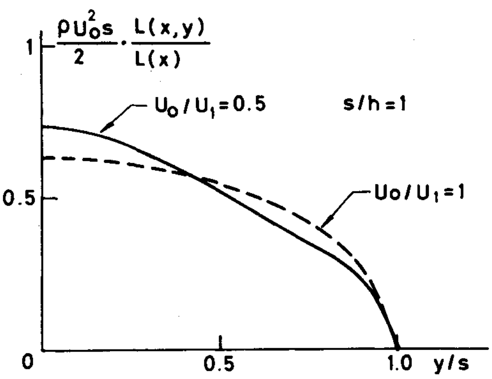


Fig. 6 Spanwise lift distribution in wake stream.

We find that the jet, wake, and the nonlinear shear profile give much larger changes of lift than the linear shear profile. This is caused mainly by the direct effect of the second derivative of stream velocity $U''(0)$ on the integral equation (20), through the logarithmic part of the kernel. Thus $U''(0)$ affects the Eqs. (26) directly through the terms involving $\lambda = U''(0)s^2/8U(0)$. It appears that a concave velocity profile ($U''(0)>0$) gives rise to a positive lift increment (Figs. 3 and 5), while a convex profile ($U''(0)<0$) produces a negative lift increment (Fig. 3).

The lift increments become stronger as the velocity ratio varies away from unity and as the scale ratio $s/h$ increases. For the jet, wake, and nonlinear shear profiles, lift increments of more than 50% are obtained in a wide range of the parameters. When $s/h$ increases, the relative lift increments tend apparently to finite asymptotic values.

The spanwise lift distribution $L(x,y)$, calculated from the expansion equation (24) for the case of a wake stream, is shown in Fig. 6. While in a uniform stream the lift distribution of a slender plane wing is elliptic ($L_1 \neq 0$, $L_m = 0$ for $m \geq 2$); in a nonuniform stream we have in general $L_m \neq 0$ so that the spanwise lift distribution is no longer elliptic.

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